Linear Algebra & Geometry

LECTURE 9

- Dimension of a vector space
- Matrices

Theorem (Replacement Lemma, Steinitz Lemma)

Suppose in some vector space V over $\mathbb{K}$ $S = \{v_1, v_2, \dots, v_n\}$ is linearly independent and $R = \{w_1, w_2, \dots, w_k\}$ spans V, i.e., $span(R) = V$. Then

1. $n \leq k$
2. There exist $i_1, i_2, \dots, i_n \in \{1,2, \dots, k\}$ such that $i_1 < i_2 < \cdots < i_n$ and $\left(R \setminus \left\{w_{i_1}, w_{i_2}, \dots, w_{i_n}\right\}\right) \cup S$ spans V.

Proof.

Since part 1. is an obvious consequence of part 2., we only need to prove part 2. This is done by induction on n.

Proof (of 2., induction on n)

The *base case*, $n = 1$.

$S = \{v_1\}$ is linearly independent iff $v_1 \neq \Theta$ (from $av = \Theta \Rightarrow a = 0 \vee v = \Theta$). We must show that one vector from R can be replaced by v_1 preserving the spanning property (that's why the thing is called *the replacement lemma*). Since R spans V, $v_1 = a_1 w_1 + a_2 w_2 + \cdots + a_k w_k$ for some $a_1, a_2, \ldots, a_k \in \mathbb{K}$. Since $v_1 \neq \Theta$, at least one coefficient is different from 0. Without loss of generality, we can say $a_1 \neq 0$. This implies that $w_1 = (a_1)^{-1} v_1 + (-a_2 a_1^{-1}) w_2 + \cdots + (-a_k a_1^{-1}) w_k$ i.e. $w_1 \in span\{v_1, w_2, \ldots w_k\}$ so, $V \supseteq span\big((R \setminus \{w_1\}) \cup \{v_1\}\big) = span\{v_1, w_2, \ldots w_k\} = span\{v_1, w_1, w_2, \ldots w_k\} \supseteq span\{w_1, w_2, \ldots w_k\} = V$ hence, $V = span\big((R \setminus \{w_1\}) \cup \{v_1\}\big)$.

QED

The *induction step*.
Suppose $\{v_1, v_2, \ldots, v_{n+1}\}$ is linearly independent, $\{w_1, w_2, \ldots, w_k\}$ spans V, and the lemma is true for every n-element linearly independent set, in particular for $\{v_1, v_2, \ldots, v_n\}$.
Hence, $k \geq n$ and some n vectors from the spanning set $\{w_1, w_2, \ldots, w_k\}$ can be replaced by $v_1, v_2, \ldots, v_n$. Without losing generality, we may assume that the replaceable vectors are $w_1, w_2, \ldots, w_n$. So, $v_{n+1} = a_1 v_1 + a_2 v_2 + \cdots + a_n v_n + a_{n+1} w_{n+1} + \cdots a_k w_k$.
Since v_{n+1} is NOT a linear combination of $v_1, v_2, \ldots, v_n$, at least one of $a_{n+1}, \ldots, a_k$, say a_t, is nonzero hence, as in the first part, w_t can be replaced by v_{n+1}. QED

Corollary.

If A and B are bases of a vector space V over $\mathbb{K}$ then $|A| = |B|$.

Proof.

Since A spans V and B is linearly independent, $|A| \geq |B|$.
Since B spans V and A is linearly independent, $|B| \geq |A|$. QED

In other words, in a finite-dimensional vector space every two bases have the same size. Hence, the following definition makes sense:

Definition.

The *dimension* of a (finite-dimensional) vector space V is the number of vectors in any of its bases.

We denote the dimension of V by $\dim(V)$.

Examples.

1. For every $\mathbb{K}^n$, $\dim(\mathbb{K}^n) = n$.
2. $\dim(\mathbb{R}_n[x]) = n + 1$.
3. $\dim(\mathbb{R}[x])$ is infinite.
4. $\dim(\mathbb{C}) = 2$ (over $\mathbb{R}$).
5. $\dim(\mathbb{C}) = 1$ (over $\mathbb{C}$).
6. $\dim\left(2^{\{a_1,a_2,\ldots,a_n\}}\right) = n$ over $\mathbb{Z}_2$.

Theorem. (6-pack theorem)

Suppose V is a vector space, $\dim(V) = n$, $n > 0$ and $S \subseteq V$.
Then

1. If $|S| = n$ and S is linearly independent, then S is a basis for V
2. If $|S| = n$ and $span(S) = V$ then S is a basis for V
3. If S is linearly independent, then S is a subset of a basis of V
4. If $span(S) = V$ then S contains a basis of V
5. S is a basis of V iff S is a maximal linearly independent subset of V
6. S is a basis of V iff S is a minimal spanning set for V.

Proof (Some parts are left as an exercise).

1. Obvious consequence of Steinitz Lemma. (Take any basis and replace ALL its vectors with vectors from S).
2. If S is not a basis, it is linearly dependent hence, a vector from S, say w is a linear combination of vectors from $S \setminus \{w\}$. But then $span(S) = span(S \setminus \{w\}) = V$, which means $S \setminus \{w\}$ is a spanning set of fewer vectors than some linearly independent set (any basis), contrary to Steinitz Lemma.
4. Easy if S is finite (*you remove, one by one, vectors who are linear combinations of other vectors from S until you get* n *vectors and then you use 2.*). In case S is infinite you can't apply Steinitz Lemma directly. Your job is to show that in a finite-dimensional vector space every infinite spanning set contains a finite spanning subset.

MATRICES

Definition.

An $m \times n$ *matrix* over a field $\mathbb{F}$ is a function

$$A: \{1,2,\dots,m\} \times \{1,2,\dots,n\} \to \mathbb{F}.$$

A matrix is usually represented by (and identified with) an $m \times n$ ("m by n") array of elements of the field (usually numbers). The horizontal lines of the array are referred to as *rows* and the vertical ones as *columns* of the matrix. The individual elements are called *entries* of the matrix.

Thus, an $m \times n$ matrix has m rows, n columns and mn entries.

If $m = n$ we call A a *square matrix*.

Matrices will be denoted by capital letters and their entries by the corresponding small letters. Thus, in case of a matrix A we will write $A(i,j) = a_{i,j}$ and will refer to $a_{i,j}$ as the element of the i-th row and j-th column of A.

On the other hand, we will use the symbol $[a_{i,j}]$ to denote the matrix A with entries $a_{i,j}$.

Rows and columns of a matrix can (and will) be considered vectors from $\mathbb{F}^n$ and $\mathbb{F}^m$, respectively, and will be denoted by $r_1, r_2, \ldots, r_m$ and $c_1, c_2, \ldots, c_n$. The expression $m \times n$ is called the *size* of the matrix.

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{bmatrix}$$

Algebra of matrices

Definition.

We add and scale matrices as we do functions.

Matrix addition is only defined for matrices of matching sizes,

$(A+B)(i,j) = A(i,j) + B(i,j)$ for every $i, j,\ 1 \leq i \leq m, 1 \leq j \leq n$ (addition of functions).

$(cA)(i,j) = cA(i,j), 1 \leq i \leq m, 1 \leq j \leq n$ (multiplication of a function by a constant)

Fact.

The set of all $m \times n$ matrices over a field $\mathbb{F}$ ($\mathbb{F}^{[m]\times[n]}$) with these operations is a vector space over $\mathbb{F}$. Its dimension is mn.

(Note: $[k] = \{1,2,\ldots,k\}$)

Proof. $\mathbb{F}^{[m]\times[n]}$ is a vectors space because the set of functions from any set into a field with component-wise operations is a vector space. Matrices $A_{p,q}$ where

$A_{p,q}(i,j) = \begin{cases} 1 \text{ for } i = p, j = q \\ 0 \qquad \text{otherwise} \end{cases}$ form a basis of $\mathbb{F}^{[m]\times[n]}$

One can easily notice that $\mathbb{F}^{m\times n}$ is isomorphic to $\mathbb{F}^{mn}$ and these matrices correspond to unit vectors of $\mathbb{F}^{mn}$. QED

Matrix multiplication is NOT defined as multiplication of functions!

Definition.
Let A be an $m \times n$ and B a $p \times q$ matrix. If $n = p$
$(AB)(i,j) = \sum_{s=1}^{n} A(i,s)B(s,j)$, for every $1 \leq i \leq m$ and $1 \leq j \leq q$.
Otherwise AB is not defined.
AB is clearly an $m \times q$ matrix.

Matrix multiplication is non-commutative, it may even happen that AB exists while BA does not.

Example (Matrix multiplication).

1. Let $A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 0 & -3 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 0 & 2 \\ -1 & 3 & 1 \\ 1 & -2 & 2 \end{bmatrix}$. Then,

   $$AB = \begin{bmatrix} 2+1+2 & 0-3-4 & 2-1+4 \\ 4+0-3 & 0+0+6 & 4+0-6 \end{bmatrix} = \begin{bmatrix} 5 & -7 & 5 \\ 1 & 6 & -2 \end{bmatrix}.$$

2. Let $A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$. Then

   $$AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } BA = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}.$$

The second example proves that AB may differ from BA even when both products exist and have the same size.

Example (Matrix multiplication).

$$\begin{bmatrix} 2 & -1 \\ 2 & 2 \\ 0 & 3 \end{bmatrix} B$$

2

4

0

$$A \begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 3 \end{bmatrix} A$$

$$B \begin{bmatrix} 2 & -1 \\ 2 & 2 \\ 0 & 3 \end{bmatrix}$$

$$X \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$A \begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 3 \end{bmatrix}$$